Transfer Function Estimation, Film Fusion and Image Restoration

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Received June 8, 1994; revised October 10, 1995; accepted October 17, 1995

The aim of this work is to restore an image from a fusion of two X-ray images. A pair of images of a same scene are obtained using two stacked films of different sensitivities. The response function of the first film presents an undesirable nonlinear shape. Nevertheless, the second image is strongly degraded by both noise and large stains, whereas the first one is not. The original problem which initiated our study was to combine these images in order to estimate a unique image free of noise and stains and expressed in the gray scale of the second film. To perform it, we must estimate the transfer function between the response functions of the two films and remove the stains of the second image. © 1996 Academic Press, Inc.

1. INTRODUCTION

We are interested in a restoration-type problem which can be formulated in the following way. Consider the pair of X-ray images shown in Fig. 4a and Fig. 4b. These images have been obtained simultaneously from two stacked films $\mathcal{F}$ and $\mathcal{P}$ of different sensitivities. Each film has different properties; some are desirable and others are not. For instance, the film $\mathcal{F}$ leads to slightly noisy images but its response function is nonlinear; conversely, the response function of the film $\mathcal{P}$ is linear, but when it is used under extreme conditions, this film leads to noisy images. Furthermore, this noisy image is also degraded by stains due to the experimental conditions. The original problem which initiated our study was to combine these images in order to estimate a unique image free of noise and stains and expressed in the gray scale of $\mathcal{P}$, that is, in a linear grayscale.

Our second example, which will be used to test our method, is the pair of images shown in Fig. 2a and Fig. 2b. This pair has been artificially created by means of a sequence of transformations detailed in Fig. 1. In this list of figures, Fig. 2a and Fig. 2b are shown in Fig. 1a and Fig. 1f, respectively. The image in Fig. 1a is supposed to not be affected by noise. The image in Fig. 1f has been obtained by rescaling the image in Fig. 1a with a nonlinear transform, say $H$, called transfer function (TF) (see Fig. 1b and Fig. 1c), then by adding stains (Fig. 1d and Fig. 1e), and finally by degrading it with white noise. In this example, although the basic response function is not supposed to be linear, the fusion problem remains the same.

Note that while $H$ is defined without taking account of the stains, and consequently before stain contamination, our problem corresponds to a gray-scale modification problem. But, when the stain effects are added, it is no longer the case since each stain produces a particular local gray-scale modification. Of course in practice, we only have the pair of images resulting from these degradations, and thus to perform an optimal fusion, we must estimate the TF and the stains. In this paper the estimation task is the basic problem. Since the fusion task removes the noise and the stain effects, it can be seen as a restoration process as it will be presented in Section 3.

Let us emphasize that the histogram modification approach briefly recalled at the beginning of Section 4 and the classic regression TF estimate described in Section 4.1 are highly inaccurate when the estimations are performed without considering the stain effects. This problem is not a classical one; we did not find any contribution about it [1–5].

The contribution of this paper is the development of a specific procedure to estimate the TF in the presence of unknown stains. This procedure is based on a regression model in which the regressor term reflects the effects of the stains.

2. PROBLEM FORMULATION

2.1. Degradation Model

Two images of a given scene are obtained from two films $\mathcal{F}$ and $\mathcal{P}$. These images are sampled using the same rectangular grid $S$ and discretized on a set $\mathcal{L}$ of intensity levels $I$. Let $(D, D')$ be such a pair of digitized images. They correspond to the same view, and, thus, there is no displacement between them. Figures 2a and 2b and Figs. 4a and 4b are two such pairs.
$D$ is assumed to be a noisy version of a theoretical image $\delta$

$$D_s = \delta_s + \varepsilon_s, \forall s \in S,$$

(1)

where $\varepsilon_s$ is a white noise with variance $\sigma^2$.

$D'$ is supposed to be much more degraded than $D$. Under our experimental conditions, the model (1) is not a realistic one for $D'$. In fact underlying widespread stains, independent of the scene, appear in the observed images (see Fig. 2f and Fig. 4f). The stains can arise from many reasons during the exposure. For instance, in our application (cf. the second experiment described in Section 6), they are due to the use of X-ray flashes under extreme experimental conditions. By adding the effects of the stains represented by a field $T$, we get the degradation model for $D'$

$$D'_s = \delta'_s + T_s + \varepsilon'_s, \forall s \in S,$$

(2)

where $\delta'_s$ is the theoretical signal and $\varepsilon'_s$ is white noise.

The stains being widespread and smooth, we assume that the variations of the stains remain locally small.

Assumption 1. There is a fixed positive number $\Delta$, such that for every sites $s, u \in S$, we have

$$\|s - u\| < \Delta \Rightarrow T_s = T_u.$$

For instance, in our application the images are of size $512 \times 512$ and $\Delta = 20$. Let us emphasize that this hypothesis will allow us to solve the estimation problem.

Furthermore, following a classical hypothesis, the variance of the noise $\varepsilon'_s$ depends on the signal intensity $\delta'_s$: $
\text{Var}(\varepsilon'_s) = \phi(\delta'_s)$, where $\phi$ is a function defined over $L$. For the sake of simplicity, our methodology will be presented in this paper in the case of constant variance:

$$\text{Var}(\varepsilon'_s) = \sigma'^2.$$

But we have used it successfully in the general version (cf. [6, 7]) where $\phi$ is unknown and must be estimated.

Finally, the image $D'$ is much noisier than $D$ (see Fig. 2b and Fig. 4b), i.e., $\sigma'^2$ is assumed to be much larger than $\sigma^2$:

$$\sigma'^2 \gg \sigma^2.$$

Note that this hypothesis is not crucial. Our approach is still valid without it, and the experimental results remain satisfactory. But this hypothesis simplifies some expressions (see Eq. (5) below) and facilitates the presentation clarity.

2.2. Transfer Function

Our study arises from a restoration problem (see Section 3), in which we must estimate the transfer function $H$ between the gray-scale functions of $\mathcal{F}$ and $\mathcal{F}'$ in order to perform the fusion of $D$ and $D'$. This transfer function is defined as follows.

Definition 1. In the absence of noise and stain, when the film $\mathcal{F}$ responds with the value $l$ at a site $s$, the film $\mathcal{F}'$ responds with the value $H(l)$ at the same site. This means (cf. (1) and (2))

$$\forall s \in X, \delta'_s = H(\delta_s).$$

Then we can rewrite (2) as

$$D'_s = H(\delta_s) + T_s + \varepsilon'_s.$$

The goal of this paper is the estimation of $H$ and $T$. Our presentation is organized as follows. In Section 3, the restoration problem is briefly described in order to illustrate our purpose. The general method to estimate $H$ is described in Section 4. This estimation is based on the estimation of a stained and smoothed version of $H$ which is introduced in Section 5. In particular, we adopt a tricubic spline function to represent this stained and smoothed function. In Section 6, experimental results concerning the TF estimation are given in artificial and real cases.

3. Restoration Context

The transfer function $H$ can be used in different ways. To illustrate our purpose, in this section we present briefly a restoration problem which requires the estimation of $H$ and $T$. In our application, the response function of the film $\mathcal{F}$ is linear and the response function of the film $\mathcal{F}'$ is nonlinear. (In fact, in the application [6, 7], we have several films $\mathcal{F}, \mathcal{F}', \mathcal{F}''$, ... with different sensitivities). Our goal is the restoration of $\delta'$. This means that the restored scene image will be given with respect to an unknown linear scale.

By considering simultaneously $D$ and $D'$, we take advantage of the following fact: the film response is strongly noisy and contaminated by stains when the response function is linear (Fig. 4b), and conversely, the response is slightly noisy when the response function is nonlinear (Fig. 4a). Thus, if we attempt to restore the scene with only $D'$, the result will be given in a linear scale but its shape will be very degraded. On the other hand, if we attempt to restore the scene with only $D$, the shape will be well described, but in a non-desirable gray scale. By taking into account $D$ and $D'$ simultaneously, the scene can be restored both in a linear scale and without strong degradation in shape.
Using Bayesian approach [8, 9], the restoration of \( \delta' \) is obtained by minimizing the functional energy

\[
U(\delta' \mid D, D') = U_1(D, D' \mid \delta') + \gamma U_2(\delta').
\]

The role of \( U_2 \) is to characterize the local regularity of \( \delta' \), and \( U_1 \) is a “fidelity” term balanced by the parameter \( \gamma \). The construction of \( U_2 \) requires prior knowledge about the expected regularity of \( \delta' \) [9–14]. Here, we only present \( U_1 \) for illustrating the use of the TF. If we suppose that \( \varepsilon \) and \( \varepsilon' \) are Gaussian white noises, from their density it follows that

\[
U_1(D, D' \mid \delta') = \sum_s [D_s - H^{-1}(\delta'_s)]^2 \frac{1}{\sigma^2} + \sum_s [D'_s - (T_s - \delta'_s)]^2 \frac{1}{\sigma'^2}.
\]

To perform this restoration, it is clear that we must estimate \( H \) and \( T \).

4. TRANSFER FUNCTION ESTIMATION

Because of the presence of the unknown stains, it is difficult to directly estimate \( H \). Let us now state a basic relation between \( D \) and \( D' \). First, from (3) we have

\[
D'_s = H(D_s - \varepsilon_s) + T_s + \varepsilon'_s.
\]

Then we get from a Taylor’s expansion of \( H \) about \( D_s \)

\[
D'_s = H(D_s) + T_s + w_s, \quad \forall s \in S,
\]

where letting \( \hat{H} \) be the derivative of \( H \), \( w_s = -\hat{H}(D_s) \varepsilon_s + \varepsilon'_s \) is a new white noise since it is a linear combination of two independent white noises. From the hypothesis \( \text{Var}(\varepsilon'_s) \gg \text{Var}(\varepsilon_s) \) follows the approximation \( \text{Var}(w_s) \approx \text{Var}(\varepsilon'_s) \). Then we have

\[
\text{Var}(w_s) \approx \sigma'^2.
\]

Remark. When there is no stain and no noise in any of the images \( D \) and \( D' \), since the TF \( H \) is an increasing function, it follows that \( D_s \geq D_s \Rightarrow D'_s \geq D'_s \). Therefore, \( H \) could be estimated using a technique called histogram modification [15] (which is usually used to perceive details in the darkest regions of images); we can rescale \( D \) so that the enhanced image has the same histogram as \( D' \). In this case the associated mapping function of the histogram modification is the TF \( H \). However, in the presence of noise, the histogram modification implies large quantization error; in our specific case, each stain produces a particular local gray-scale modification. Therefore, the histogram modification technique is no longer valid.

To introduce our approach of estimation, let us consider another simple estimation method when there is no stain, i.e., \( T = 0 \). In the presence of stains, our method will be an extension of this simple one.

4.1. Estimation of \( H \) in the Absence of Stain: Reducing Noise and Fitting Parametric Model

When there is no stain, for all \( l \in \mathcal{L} \) and \( u \in S \) such that \( D_u = l \), it follows from (4) that \( D'_u \) is equal to \( H(l) \) degraded by a large noise \( w_u \). Naturally, to reduce the noise with respect to \( H(l) \), we take the mean of \( D'_u \) for all \( u \) satisfying \( D_u = l \)

\[
m_l = \frac{1}{|N(l)|} \sum_{u \in N(l)} D'_u
\]

\[
= H(l) + \frac{1}{|N(l)|} \sum_{u \in N(l)} w_u,
\]

where \( N(l) = \{u \in S : D_u = l \} \), and \( |N(l)| \) is the number of elements in \( N(l) \).

For each \( l \), thus compute a value \( m_l \), which is a noisy version of \( H(l) \) with a small noise \( (1/|N(l)|) \sum_{u \in N(l)} w_u \) of variance \( \sigma'^2/|N(l)| \). \( m_l \), \( l \in \mathcal{L} \) is the relevant data for \( H \), extracted from \( (D, D') \). Then the estimation of \( H \) consists in representing \( H \) by a parametric curve (a cubic spline function, for example), and in fitting it to \( \{m_l\} \) according to a least-squares criterion.

4.2. Estimation of Degraded Transfer Function \((H + T):\) Extraction of Relevant Data

In our case, because of the stains, the mean computed in (6) is no longer able to approximately represent \( H \). However, since the stains are locally constant (cf. Assumption 1), the idea consists in locally considering the previous definition (6) at every site \( s \). Denote by \( m_s = (m_{s,l}, l \in \mathcal{L}) \) the values \( (m_l, l \in \mathcal{L}) \) computed as in (6) but in a restricted small neighborhood of \( s \); then \( m_{s,l} \) is approximately a noisy version of \( H(l) + T_s \).

Let us detail the extraction of \( m_{s,j} \). Fix a site \( s \in S \) and denote by \( W_s \) a square small window centered on \( s \). For every \( l \in \mathcal{L} \) denote \( N_i(l) = \{u \in W_s : D_u = l \} \), which is the definition of \( N(l) \) restricted to \( W_s \). Then, as a local definition of (6), we consider

\[
m_{s,l} = \frac{1}{|N_i(l)|} \sum_{u \in N_i(l)} D'_u.
\]

4.2.1. Decomposition of \( m_{s,l} \) and Local Window \( W_s \)

If we denote by \( \overline{T}_i(l) \) the mean of the stains \( T_u \) for \( u \in N_i(l) \), and by \( \overline{w}_{s,l} \) the mean of the noises \( w_u \) for \( u \in N_i(l) \),
$$\bar{T}_s(l) = \frac{1}{|N_s(l)|} \sum_{u \in N_s(l)} T_u,$$

$$\bar{w}_{s,l} = \frac{1}{|N_s(l)|} \sum_{u \in N_s(l)} w_u,$$

we can decompose $m_{s,l}$ as

$$m_{s,l} = H(l) + \bar{T}_s(l) + \bar{w}_{s,l}. \quad (9)$$

Denote by $(x_s, y_s)$ the coordinates of a site $s$ in $S$. The principal part $H(l) + \bar{T}_s(l)$ in Eq. (9) is a function of the three variables $x_s$, $y_s$, and $l$, called stained transfer function (STF) $G$:

$$G(x_s, y_s, l) = H(l) + \bar{T}_s(l). \quad (10)$$

We can then rewrite (9) as

$$m_{s,l} = G(x_s, y_s, l) + \bar{w}_{s,l}. \quad (11)$$

The variance of the noise $\bar{w}_{s,l}$ is $\sigma^2/|N_s(l)|$. (It must be noted that $(\bar{w}_{s,l})$ is not a white noise due to the overlapping of windows $W_s$. Given a window $W_s$, a gray level $l$ may be rare in $W_s$, or it may even happen that $l$ is absent from $W_s$. In order to overcome this problem and for another reason arising from the estimation of $G$ (see Section 5), we impose that $m_{s,l}$ have the same precision for all $s$; i.e., $\text{Var}(\bar{w}_{s,l})$ is independent of $s$, say $|N_s(l)| = c(l)$. (We choose $c(l)$ to be proportional to the histogram value $h(l)$ of $D$.) To realize it, at the site $s$, for each gray level $l$, we extend or reduce the window $W_s$, such that $|N_s(l)|$ is close to $c(l)$. Consequently, we have

$$\text{Var}(\bar{w}_{s,l}) = \frac{\sigma^2}{c(l)}. \quad (12)$$

4.2.2. Localization Property of $\bar{T}_s(l)$

Finally, we must examine how $m_{s,l}$ is relevant for $H(l) + T_s$. From (9), $m_{s,l}$ is a noisy version of $H(l) + T_s$ if $\bar{T}_s(l) = T_s$. There are two cases. First, when a given gray level $l$ is very often observed around $s$ in $D$, then the window $W_s$ is of size smaller than $\Delta$ (i.e., $N_s(l)$ is contained in the window of size $\Delta$ centered on $s$). In this case, if $u \in N_s(l)$, then $T_u = T_s$ (cf. Assumption 1), and the mean $\bar{T}_s(l)$ is close to $T_s$. Second, at the opposite, when a gray level $l$ is rare around $s$, $N_s(l)$ may contain sites which are too far from $s$, then $\bar{T}_s(l)$ and $T_s$ are rather different. To discriminate between these two cases, we need to set the following definition.

Definition 2. For all $s \in S$, we denote by $\mathcal{L}$ the set of gray levels $l \in \mathcal{L}$ such that the corresponding windows $W_s$ are of sizes smaller than $\Delta$.

An immediate consequence of the previous definition is

$$\text{for all } l \in \mathcal{L}, \bar{T}_s(l) = T_s. \quad (13)$$

It must be noted that Eqs. (10) and (13) are crucial for the estimation of $H$ and $T$. This estimation is described below, assuming that $G$ has been estimated.

4.3. Estimation of Transfer Function $H$

Suppose that $G$ is known. In fact, $G$ is replaced by its estimate $\hat{G}$ obtained in Section 5.

$G$ is defined by Eq. (10) in which the mean $\bar{T}_s(l)$ is given by (8). If $l \in \mathcal{L}$, the mean $\bar{T}_s(l)$ is close to $T_s$ (cf. (13)). Hence, we obtain a system of linear equations linking the variables $\{H(l), l \in \mathcal{L}\}$ and $\{T_s, s \in S\}$ with $G$

$$\begin{cases}
(i) H(l) + \frac{1}{|N_s(l)|} \sum_{u \in N_s(l)} T_u = G(x_s, y_s, l), & \forall s \in S \text{ and } l \in \mathcal{L} \setminus \mathcal{L}_s \\
(ii) H(l) + T_s = G(x_s, y_s, l), & \forall s \in S \text{ and } l \in \mathcal{L}_s,
\end{cases} \quad (14)$$

where $\mathcal{L} \setminus \mathcal{L}_s$ is the set of the gray levels that are not in $\mathcal{L}_s$. We must solve this system to determine the values $H(l)$ and $T_s$. However, this system contains $|\mathcal{L}| \times |S|$ equations with only $|\mathcal{L}| + |S|$ variables. There are much more equations than variables, and for this reason it is natural to get the least-squares solution of the system.

The problem of estimating $H$ is defined up to an additive constant. In fact, if $(\hat{H}, \hat{T})$ is a solution of (14), then for any constant $b$, $(\hat{H} + b, \hat{T} - b)$ is also a solution of the system. Therefore, we use a constraint on $H$ like $H(1) = a$, where $a$ is a fixed known value.

Since the computation is expensive, we propose to reduce the variable space dimension in order to simplify the resolution. The idea is to replace $T$ in the equations with expressions using $H$ and $G$ which are deduced from equations (ii). The solution is detailed in the Appendix.

Remark. Assumption 1, which leads to equations (ii) of system (14), cannot only be considered as a condition which allows us to simplify computation by reducing the variable space dimension, but also as a determinant information permitting us to distinguish the stains from degraded signals. Let us detail this point. Without Assumption 1, $\mathcal{L}_s$ is empty, and the system (14) is formed only by equations (i). For any arbitrary function $\bar{H}$ defined over $\mathcal{L}$, suppose that $\bar{T}_u = \delta_u + \bar{T}_u - \bar{H}(D_s)$. Hence $(\bar{H}, \bar{T})$ is a solution of the system. Indeed, in this case, $\bar{H}(l) +
1/|N_t(l)| \sum_{u \in N_t(l)} T_u = 1/|N_t(l)| \sum_{w \in N_t(\delta')} (\delta'_u + T_u) = G(x_s, y_s, l). Therefore, we cannot in any way determine H and T from the system.

5. ESTIMATION OF STAINED TRANSFER FUNCTION G

Consider the model (11) \( m_{i,j} = G(x_s, y_s, l) + \bar{w}_{i,j} \). Since G is the result of a smoothing procedure, it is natural to represent it by a smooth function, that is, a tricubic spline on \( S \times L \). In the framework of spline regression [16–18], the data grid is partitioned by setting equidistant nodes on each coordinate x, y, and l. Let \( \mathcal{N}_1, \mathcal{N}_2, \) and \( \mathcal{N}_3 \) be these node sets and \( B^1, B^2, \) and \( B^3 \) the 1D B-spline function sets built on each of them. G is modeled by a tricubic spline over \( S \times L \) with node set \( \mathcal{N}_1 \times \mathcal{N}_2 \times \mathcal{N}_3 \). Its representation is

\[
G(x, y, l) = \sum_{i,j,k} B^1_i(x) B^2_j(y) B^3_k(l) \alpha_{i,j,k},
\]

where \( \alpha_{i,j,k} \) are the parameters of the model. We are led to estimate G by optimizing a weighted least-squares criterion [20] with respect to \( \alpha \) by computing

\[
\hat{\alpha} = \arg \min_\alpha \sum_{i,j} \left( m_{i,j} - G(x_s, y_s, l) \right)^2 \frac{1}{\text{Var}(\bar{w}_{i,j})}.
\]

Of course \( \hat{\alpha} \) cannot be directly computed in its vector form since its size is extremely large (in our application, we have placed seven nodes in each coordinate of \( S \times L \); there are therefore 1331 parameters). Yet, due to the separability of the B-splines in (15), when \( \text{Var}(\bar{w}_{i,j}) \) is constant, it is well known that \( \hat{\alpha} \) can be calculated in a simple way [17, 18] by considering \( m, G, \) and \( \alpha \) as a three-dimensional matrix. As described in [6, 7], an extension is the case where the variance is not constant but separable; that is, 
\( \text{Var}(\bar{w}_{i,j}) = p_x q_y r_l \). It is our case with \( p_x = 1, q_y = 1, \) and \( r_l = \sigma^2/c(l) \). The need of this separability of the variance explains also why \( |N_t(l)| \) has been chosen independently of \( s, \) in (12).

6. EXPERIMENTAL RESULTS

(1) First, in order to illustrate all the concepts introduced in this paper, we experiment with an artificial data set \((D, D')\). The original image \( \delta \) is the well-known picture “Lena” often used in image processing experiments (Fig. 1a). In this first experiment, we took \( D \) equal to \( \delta \), and we digitized it on 160 gray levels. \( H \) was chosen as

\[
H(l) = c_1 \frac{l^d}{(c_2 + l^d)} + c_3,
\]

with \( c_1 = 270, c_2 = 270/2, c_3 = 20, \) and \( d = 3.5, \) (see Fig. 1b). Figure 1c shows \( \delta' = H(\delta) \). The stain field \( T \) was obtained by summing two nonnormalized Gaussian surfaces, their modes being equal to half the gray-level range of \( \delta' \) (Fig. 1d). The stained image is shown in Fig. 1e. \( \varepsilon' \) is a white noise field of variance \( \sigma^2 = 5800. \) \( D' \) has been artificially created using the degradation models \( D' = \delta' + T + \varepsilon' \) (Fig. 1f). Note how strong these degradations are.

For computing the data \( m_{i,j} \), instead of working with the whole set \( S \), we restrict the treatment to a coarser rectangular grid in order to reduce computing time. The spline representation of \( G \) and \( H \) is done by placing seven interior nodes on each coordinate of \( S \) and \( L \). The TF \( H \) has been estimated as described in the previous sections. \( H \) and its estimate \( \hat{H} \) are presented in Fig. 1b; they are nearly the same. These stain effects are emphasized in Fig. 3a and Fig. 3b where we show \( \{G(x_s, y_s, l), l \in L \} \) for two different sites. The image \( \hat{H}(D) = \{\hat{H}(D_s), s \in S \} \) and the estimate \( \hat{T} \) of the stains are given in Fig. 2c and Fig. 2d. We note that the images \( \hat{H}(D) \) and \( H(D) \) (in Fig. 1c) are nearly the same numerically, as well as \( \hat{T} \) and \( T \). This shows the efficiency of our method. Finally, Fig. 3c shows the \( H \) estimate obtained using a classic estimation presented in Section 4.1, and which does not take into account the stain contamination.

(2) Second, we are concerned with a nondestructive control task which has motivated this paper. Let us describe briefly the context of our application. The images \( D \) and \( D' \) are X-ray images of the same object, (Fig. 4a and Fig. 4b). Note that the degradation of \( D' \) is mainly the result of extreme experimental conditions. In fact, the radiography is carried out in a few nanoseconds and with high energy (about 14 MeV). Due to the nature of the object and the short exposure time, the transmitted signal is weak. Physicists use films with high sensitivity associated to the amplifier. These conditions lead to the problem addressed in this paper. Our images \( D \) and \( D' \) are treated by the restoration-fusion algorithm described above. This treatment is repeated for a limited number of projections and the resulting restored images are used to carry out a tomographic reconstruction of the object [21]. This reconstruction goal is the reason why we search for an image restoration which yields a high precision for each restored pixel value, and more precisely in a linear scale.

Let us present now the result of our method on these images. Here, we must restore an axial object. Figure 4c, as Fig. 2c, shows the image \( \hat{H}(D) \). It does not contain any
FIG. 1. Construction of an artificial pair \((D, D')\). (a) Image \(D = \delta\). (b) Transfer function \(H\) and its estimate. (c) Theoretical image \(H(\delta)\). (d) Image of the stains \(T\). (e) Stained image \(H(\delta) + T\). (f) Stained and noisy image \(D' = H(\delta) + T + \varepsilon'\).
more stains; our procedure has removed them. Figure 4d, as Fig. 2d, shows the estimated stains $\hat{T}$, as we can guess in Fig. 4b.

7. CONCLUSION

The fusion of a pair of stacked images, one of which being strongly degraded by noise and unknown stains, is studied. The main goal of this paper is the estimation of the transfer function between the sensitivity curves of the films from which the images were digitized, and the estimation of the stains. Classic methods are highly inaccurate because of the stains. For this, we have defined a useful function reflecting the effects of stains, called the stained transfer function. From the estimate of the STF obtained using a tricubic spline representation, we can establish a system of linear equations in which the unknown variables are the TF and the stains. A feasible method to get a least-squares solution in reducing the variable space dimension of the system is given. As a result of this estimate, the fusion process can provide satisfactory image restoration. Note that generalization to a fusion of more than two films is trivial and is simply a matter of notation.

APPENDIX

Solving the System in Reducing the Space Dimension

For the sake of simplicity, we suppose $\mathcal{L} = \{1, 2, 3, \ldots, n\}$, and we denote by $H$ the column vector whose components are $\{H(l)\}$. From (ii), for every site $i \in S$ and every $l \in \mathcal{L}$, we have $T_i = G(x_i, y_i, l) - H(l)$. But if we replace $T_i$ in every equation of the system by $G(x_i, y_i, l) - H(l)$, the system will become a trivial one without any variable. To get meaningful equations, we prefer to take the average of the $|\mathcal{L}|$ equations above.
FIG. 3. (a) $\hat{G}(x_s, y_s, \cdot)$ at a particular site $s$. (b) $\hat{G}(x_t, y_t, \cdot)$ at another site $t$. (c) A classic estimate of $H$.

$$T_i = \frac{1}{|\mathcal{A}|} \sum_{l \in \mathcal{A}} G(x_i, y_i, l) - \frac{1}{|\mathcal{A}|} \sum_{l \in \mathcal{A}} H(l)$$

$$= \overline{G}_i - V_i H,$$

where $\overline{G}_i = (1/|\mathcal{A}|) \sum_{l \in \mathcal{A}} G(x_i, y_i, l)$, $V_i$ is the line vector whose $l$th component is $1/|\mathcal{A}|$ if $l \in \mathcal{A}$, and 0 otherwise.

We then replace $T$ in (14) by the right hand of (16). At this stage, the system has only $|\mathcal{A}|$ variables $\{H(l)\}$, and can be written as

$$X_s H = Y_s, \forall s \in S,$$

where, letting $I_l$ be the $l$th line of the unit matrix of order $|\mathcal{A}|$, for $l \in \mathcal{A} \setminus \mathcal{L}$, the $l$th element of the vector $Y_s$ is equal to $G(x_s, y_s, l) - (1/|\mathcal{N}_s(l)|) \sum_{u \in \mathcal{N}_s(l)} \overline{G}_u$, and the $l$th line of the matrix $X_s$ is equal to $I_l - (1/|\mathcal{N}_s(l)|) \sum_{u \in \mathcal{N}_s(l)} V_u$; for $l \in \mathcal{L}$, the $l$th element of $Y_s$ is equal to $G(x_s, y_s, l) - \overline{G}_s$, and the $l$th line of $X_s$ is equal to $I_l - V_s$.

We can still reduce the space dimension of the variables $\{H(l)\}$ using the spline representation $H = B\hat{\beta}$, where $B$ is a matrix constructed on a set of cubic spline functions, $\hat{\beta}$ is the parameter vector. Hence, $\hat{\beta}$ is calculated by computing

$$\hat{\beta} = \arg \min_{\beta} \sum_{s} \| X_s B\beta - Y_s \|^2$$

under the constraint $H(1) = a$. Finally, $H$ is estimated by $\hat{H} = B\hat{\beta}$. This criterion allows both the fitting of $H$ to the data $Y_s$ and the smoothing of $H$, contrary to $\min_{H} \sum_{s} \| X_s H - Y_s \|^2$. Now, recalling (16), an estimate of the stains is

$$\hat{T}_s = \frac{1}{|\mathcal{A}|} \sum_{l \in \mathcal{A}} G(x_s, y_s, l) - \frac{1}{|\mathcal{A}|} \sum_{l \in \mathcal{A}} \hat{H}(l).$$

Note that $\hat{T}$ estimated above is also the least-squares solution of equations (ii) of the system.


ACKNOWLEDGMENTS

The authors acknowledge Commissariat à l'Energie Atomique (CEA) for the financial support provided to this study. This work was also supported by SUDIMAGE group.

REFERENCES

1. R. L. Whitman, Combination of several radiographic film types, private communication, Los Alamos National Laboratory, 1990.