3D Curve Reconstruction from Degraded Projections

B. Chalmond, F. Coldefy and B. Lavayssière

Abstract. Our purpose is the reconstruction of a 3D curve from a limited number of projections. These projections are highly blurred. The curve is represented by a parameterized spline function. The estimation of the spline parameters is done in the context of hidden data. In this framework we build an iterative algorithm which gives a new viewpoint of the "active curve" methods developed in image analysis.

§1. Introduction

The data are composed of a small number $L$ of projections of a 3D unknown curve, say $C^*$, ($L < 10$ in our application). The projection images are very degraded: each observed projection of the curve $C^*$ is not a binary image of a 2D curve, but is a grey level image digitized on a square grid $S^t$. Such an image can be seen as a blurred projection, or in other words as a plane having discontinuities represented by an uneven valley in which the theoretical curve projection of $C^*$ lies approximately in the bottom, [1]. Furthermore due to the degradation, the valley may be the union of several valleys. These images denoted by $g = \{g^\ell, \ell = 1, ..., L\}$ are assumed to verify $E(g^s_\ell) < 0$ if the site $s \in S^t$ belongs to a valley and $E(g^s_\ell) = 0$ otherwise, $E$ standing for the mathematical expectation. Note that we do not assume any knowledge about the physical degradation system. We denote $\{P^\ell, \ell = 1, ..., L\}$ the geometrical projection functions on the image planes. For instance, in our application each $P^\ell$ is defined from a point source $O^\ell$ which illuminates the curves $C^*$ with a 3D fan-beam of $\gamma$-rays. Denoting $\tilde{S}^t$ the projection plane associated to the source $O^\ell$, for every point $c \in C^*$ the non-degraded projection is simply $P^\ell(c) = \tilde{S}^t \cap \tilde{O}^\ell c$, where $\tilde{O}^\ell c$ is the ray emitted by $O^\ell$ and passing through $c$. 

Wavelets, Images, and Surface Fitting

Copyright © 1994 by AK Peters, Boston.
ISBN 0-12-XXXX.
All rights of reproduction in any form reserved.
Our goal is to determine a 3D curve, say $\mathcal{C}$, whose projections $\mathcal{P}^i(\mathcal{C})$ are coherent with the degraded projections of $\mathcal{C}^*$, that is with the valleys. This is an attempt to reconstruct $\mathcal{C}^*$, but $L$ being too small, it is illusory to hope an accurate reconstruction. Hence we shall only seek a curve which presents the following characteristics of $\mathcal{C}^*$: its situation in the space, its orientation and its general shape. In our presentation, we shall not speak about the technical difficulties arising from the ordering of the discretized curve points.

§2. The curve model

Our approach is related to the "active curve" method, mainly developed in image analysis research. In the two dimensional case, a model curve is placed on the image and is subject to the action of "external forces" which move and deform it, to best fit it to the desired features in the image. Formulations of problems of this kind are mostly deterministic and involve solutions to variational principles [2,3]. Stochastic formulations based on probabilistic models have also been proposed [4]. If $\omega$ denotes the generic curve, these approaches consist in constructing an energy function $U(\omega, g)$ which tells us how much is plausible any curve $\omega$ given the data $g$, and then to compute iteratively $\min_\omega U(\omega, g)$, starting with an initial curve $\omega^0$. In fact in most of the cases, the active curve $\omega^k$ obtained at each iteration $k$ converges to a local minimum which is highly dependent on $\omega^0$: the curve is attracted by the nearest features around $\omega^0$ in $g$. Then, the practical strategy consists in choosing $\omega^0$ in the vicinity of the desired features. Here, we want to alleviate this drawback since we know there is only one curve to reconstruct.

We search for a curve represented by $\mathcal{C} = \omega + \delta$ where $\omega$ is a smooth curve and $\omega + \delta$ is a deformed version of $\omega$ by the deformation $\delta$. $\omega$ is a deterministic global representation. As in [5], each coordinate $x, y, z$ of $\omega$ is a cubic spline function on $[0, 1]$: \[
\omega(u) = (x(u), y(u), z(u)) = \sum_{j=1}^{m} B_j(u) \theta_j,
\]
where $u \in [0, 1]$ and $\theta_j = (\theta_j^x, \theta_j^y, \theta_j^z)$ are the spline parameters, $m$ is the number of knots defining the spline function on $[0, 1]$ and $B_j$ is the associated B-spline basis. Pratically, the curve $\omega$ is discretized into $\{\omega_i = \omega(u_i); i = 1, ..., n\}$, $m < n$. The fraction $n/m$ defines the smoothing power of the spline. Denoting by $B_j^i = B_j(u_i)$ the generic element of the B-spline matrix for this discretization, we have $\omega_i = \sum_{j=1}^{m} B_j^i \theta_j$, and finally the matrix form of the discrete curve \[
\omega = B \theta,
\]
where now $\omega$ is a matrix of size $(n, 3)$. $\delta$ is a stochastic local representation. $\omega_i + \delta_i$ is seen as a random deformation of $\omega_i$ by $\delta_i$, according to a Gibbs distribution, that will be described below. The global representation captures the general shape whereas the local one reconstructs details that the global abstraction misses.
§3. The energy without $\delta$

Let $\omega^* = B\theta^*$ be the true unknown global curve. $\theta^*$ is unknown and is the main parameter to estimate. If "noisy" discretized curve versions of $P^t(\omega^*)$ were given, say $\xi^t$, then an estimation of $\theta^*$ could be

$$\hat{\theta} = \arg \min_{\theta} \sum_t \| \xi^t - P^t(B\theta) \|^2. \quad (1)$$

Here, the difficulty arises from the fact that $\xi$ cannot be directly observed.

For every $n$-polygonal line $\omega$, (not necessary of spline type), we define the log-likelihood of $\theta$ with respect to $(\omega, g)$ by

$$\log P_\theta(\omega \mid g) \propto -U_\theta(\omega, g)$$

$$U_\theta(\omega, g) = \sum_{t=1}^L U^t_\theta(\mathcal{P}^t(\omega), g^t),$$

where $P_\theta$ denotes a probability. The energies $\{U^t_\theta\}$ are dependent because of the projections $\mathcal{P}^t$ which are applied to the same $\omega$. They are defined by

$$U^t_\theta(\mathcal{P}^t(\omega), g^t) = \sum_{i=1}^n \| \mathcal{P}^t(\omega_i) - \mathcal{P}^t(B_i\theta) \|^2 + \sum_{i=1}^n \| s^t_i(\omega, g) - \mathcal{P}^t(B_i\theta) \|^2,$$

where $s^t_i(\omega, g) \in \mathcal{S}^t$ is called "global attractor site" of $\mathcal{P}^t(\omega_i)$ and is computed from $g^t$. Its definition is application-dependant, and is rather difficult to design. At this point, any prior information concerning the characteristics of the degradation can be taken into account. To give a glimpse of it, one can consider the simplest definition, that is the barycenter

$$s^t_i(\omega, g) \propto \sum_{j \in \mathcal{A}_i^t} g^t_j j, \quad (2)$$

where $\mathcal{A}_i^t$ is the set of sites in the projection plane for which $\mathcal{P}^t(\omega_i)$ is the nearest point of the projected curve: $j \in \mathcal{A}_i^t$ if $i = \arg \min_i \| j - \mathcal{P}^t(\omega_i) \|$. Note that every site in $g^t$ contributes to the definition of these attractors and this energy favours the bottom valleys.

§4. The algorithm without $\delta$

The question now is how to estimate $\theta^*$? We place this estimation problem in the context of hidden data taking $g$ as the observed data and $\omega$ as the hidden data. This terminology is meaningful when one tries to estimate $\theta^*$ by the maximum likelihood principle. The interested reader can refer to [6] for
a general presentation of this topic. Since \( \omega \) is hidden, the maximum likelihood principle is not directly applicable, and thus we propose the following iterative algorithm. Let \( \Omega \) denote the random curve whose \( \omega \) is the occurrence. At step \((k + 1)\), the estimate of \( \theta^* \) is:

\[
\theta^{k+1} = \arg\min_{\theta} E[U_\theta(\Omega, g) \mid \theta^k],
\]

where the mathematical expectation \( E[.] \) is defined by

\[
E[U_\theta(\Omega, g) \mid \theta^k] = \sum_\omega U_\theta(\omega, g)P_{\theta^k}(\omega).
\]

We choose the simplest expression for the probability distribution \( P_{\theta^k} \):

\[
P_{\theta^k}(\omega) = I(\omega = B\theta^k), \tag{3}
\]

where \( I \) is the indicator function. In this case, it follows \( E[U_\theta(\Omega, g) \mid \theta^k] = U_\theta(\omega^k, g) \). Thus, at the step \((k + 1)\), the solution is:

\[
\theta^{k+1} = \arg\min_{\theta} U_\theta(\omega^k, g), \tag{4}
\]

\[
\omega^{k+1} = B\theta^{k+1}.
\]

At each step \( k \), note that the "observations" \( \xi^l \) introduced in (1), can be indentified with \((P^l(\omega^k), s^l(\omega^k, g))\). In (4), we have to solve a non-linear least-square problem. In our application, if \((x, y, z)\) denotes the coordinate system with the source \( O^l \) at position \((O^l_x, O^l_y, O^l_z)\) and if the projection plane \( \tilde{S}^l \) is horizontal at \( z = \kappa \), it is straigtforward to prove that

\[
P^l_x(x, y, z, \theta) = \frac{B_x \theta^x - O^l_x \theta^\kappa - O^l_z \theta^\kappa}{B_x \theta^z - O^l_z \theta^\kappa} \theta^z + O^l_z,
\]

and with a similar expression for the \( y \) coordinate. The convergence of this deterministic algorithm is guaranted since at each iteration the log-likelihood decreases, and it is very fast. Let us emphasize that the serious weakness of this kind of approach is due to the use of the attractor sites, a difficulty which is also present in the other active curve methods when we have to express the "external forces" energy.

§5. The algorithm with \( \delta \)

The role of the previous energy is to determine a global representation of the curve taking account of the whole data set \( g \). But such a representation must be also stable with respect to the local data surrounding the curve. To this end, we rewritte the energy as \( U_\theta(\omega + \delta, g) \) where \( \delta \) is included in order to determine a curve \( C = \omega + \delta \) which verifies this stability. On \( \delta \), we define a probability distribution \( P(\delta \mid \omega, g) \) which characterizes the local environment.
and then the energy is iteratively minimized according to a Bayesian approach based on this distribution.

The hidden data now are composed of ω and δ. Let Δ denote the random deformation whose δ is the occurrence. At the step \((k + 1)\), the estimate of \(\theta^*\) is:

\[
\theta^{k+1} = \arg\min_\theta E[U_\theta(\Omega + \Delta, g) \mid \theta^k].
\]

The mathematical expectation \(E[\cdot]\) is defined by

\[
E[U_\theta(\Omega + \Delta, g) \mid \theta^k] = \sum_{\omega, \delta} U_\theta(\omega + \delta, g)P(\delta \mid \omega, g)P_{\theta^k}(\omega)
= \sum_{\delta} U_\theta(\omega^k + \delta, g)P(\delta \mid \omega^k, g),
\]

where the distribution (3) has been again considered for \(P_{\theta^k}\). In this new expression, note that the model \(B\theta\) concerns \(\omega^k + \delta\). We define the following Gibbs distribution on \(\delta\):

\[
P(\delta \mid \omega^k, g) \propto P(\delta) P(g \mid \omega^k + \delta)
\propto \exp - \sum_i \| \delta_i - \delta_{i-1} \|^2
\exp - \lambda \sum_i \sum_{\ell} \| \tilde{s}^\ell_i(\omega^k, g) - \mathcal{P}^\ell(\omega^k) \|^2.
\]

where \(\tilde{s}^\ell_i(\omega^k, g) \in S^\ell\) is called "local attractor site" of \(\mathcal{P}^\ell(\omega^k)\). The global attractor sites are computed by visiting all the sites in \(S^\ell\) and looking for the nearest point on \(\omega^k\) for any such site. Conversely, for the local attractor sites, one move on the curve \(\omega^k\), and for every site \(i \in \omega^k\) one look for the nearest valley sites in a neighborhood \(\mathcal{N}^\ell_i\) of this site. As in (2), the simplest definition is the barycenter

\[
s^\ell_i(\omega, g) \propto \sum_{j \in \mathcal{N}^\ell_i} g^\ell_j j.
\]

\(\lambda\) is a weighting parameter which balances the influence of the two terms in the Gibbs distribution, and is determined following heuristic considerations. The sum (5) is computed using a Monte Carlo method that simulates a finite set of configurations \(\delta\) according to the Gibbs distribution, [4,6]. It would remain to give a scheme for the chaining of the iterations. Unfortunately, we have no grounded result concerning the convergence of this second algorithm that we think to be depending on the scheme type. In our application, the reconstruction is a two-stage procedure. First it is started by using the first algorithm and then is finished by the second one but with \(\omega\) fixed to the configuration obtained at the first stage.
§6. Experimental results

In the context of our application, we now illustrate the method on $\gamma$-radiographies. Figure 1 presents the degraded projections respectively obtained from $L = 3$ aligned $\gamma$-sources. If $G$ denotes the barycenter of $C^*$, then the sources are placed in such a way that their angles approximatively satisfy $(\overline{O^1G}, \overline{O^2G}) = (\overline{O^2G}, \overline{O^3G}) = 20^\circ$. Furthermore, the projection images belong to the same plane. We have run our two-stage algorithm, starting with an initial curve $\omega^0$ randomly drawn in the 3D space. The first stage needs twenty iterations. Figure 1 shows also how the projections of the reconstructed curve at different iterations, are coherent with the degraded projections. The first row of Figure 1 shows the projections $\{\mathcal{P}^f(\omega^0)\}$. The second row depicts $\{\mathcal{P}^f(\omega^{10})\}$ and the third one gives a $\{\mathcal{P}^f(C)\}$ set obtained at convergence.

![Projections of the curve $\omega^0$](image1)

![Projections of the curve $\omega^{10}$](image2)

![Projections of the curve at convergence](image3)

**Figure 1.** Reconstruction with $L = 3$
Acknowledgments. This work was supported by Electricité De France (DER-SDM) and SUDIMAGE (Automatic Imaging).

References


Bernard Chalmond
Cergy-Pontoise University
8, Le Campus
95033 Cergy-Pontoise Cedex, France
chalmond@diam1.ens.cachan.fr

F. Coldefy is with SUDIMAGE group, Cachan, France.

B. Lavayssière is with the DER-SDM Department,
EDF, Chatou, France.